

# Inclusion with Repetitions and Boolean Constants – Implication Problems Revisited

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## Abstract

Inclusion dependencies form one of the most widely used classes of database dependencies. We expand existing results on the axiomatization and computational complexity of their implication problem to two extended variants. First, we present an alternative completeness proof for standard inclusion dependencies and generalize it to inclusion dependencies with repetitions that can express equalities between attributes. The proof uses only two values, enabling us to work within the Boolean setting. Furthermore, we study inclusion dependencies with Boolean constants, provide a complete axiomatization, and show that no such axiomatization is  $k$ -ary. We also establish that the decision problems for both extended versions remain PSPACE-complete. The extended inclusion dependencies are common in team semantics, which serves as the formal framework for the results.

## 1 Background

Evaluating inclusion dependencies in a uni-relation database is analogous to evaluating inclusion atoms in teams, i.e., sets of assignments. In (Häggblom 2025), we re-examine the axiomatization of their implication problem, that is, finding a system consisting of rules that can solve whether one inclusion atom follows from a set of inclusion atoms.

A team  $T$  is a set of assignments  $s : \mathcal{V} \rightarrow M$ , where  $\mathcal{V}$  is a set of variables and  $M$  is any set with elements we call values. In the Boolean setting,  $M$  is the set  $\{0, 1\}$  and for convenience, we extend the scope of assignments to the Boolean constants  $\top$  and  $\perp$  such that for any assignment  $s$ ,  $s(\top) = 1$  and  $s(\perp) = 0$ . We write  $a_i, b_i, \dots$  for individual variables (either first-order or Boolean) and  $a, b, \dots$  for finite (possibly empty) sequences of variables. Let  $a$  denote  $a_1 \dots a_n$  and  $b$  denote  $b_1 \dots b_n$ . We reserve  $x$  for a sequence of constants  $\top$  and  $\perp$ . Inclusion atoms are of the form  $a \subseteq b$ , with the semantic clause

$$T \models a \subseteq b \iff \text{for all } s \in T \text{ there exists } s' \in T \text{ with} \\ \langle s(a_1), \dots, s(a_n) \rangle = \langle s'(b_1), \dots, s'(b_n) \rangle.$$

For teams in the Boolean setting, the entailment  $\Sigma \models \alpha$  holds whenever each team that satisfies all atoms in  $\Sigma$  also satisfies  $\alpha$ .

The implication problem for standard inclusion dependencies is axiomatized in (Casanova, Fagin, and Papadimitriou 1984), under both a semantic restriction (that presupposes the existence of more than two available values) and

a syntactic restriction on repeated variables within the sequences  $a$  and  $b$ . The semantic restriction prevents the completeness result from carrying over to the Boolean setting; we address this by providing an alternative completeness proof for the same system using only two values. Dropping the syntactic restriction yields inclusion atoms that allow repetition of variables, enabling us to express equalities between the variables  $a_1$  and  $a_2$  by the atom  $a_1 a_2 \subseteq a_1 a_1$ . For this richer syntax, we prove completeness in the Boolean setting for a fragment of the system in (Mitchell 1983), which also implies completeness in the first-order setting where more values are available. Our result clarifies the behaviour of such inclusion atoms common in the literature on first-order team semantics (Galliani 2012).

We further extend the syntax by allowing Boolean constants in the variable sequences, obtaining the atoms used in propositional inclusion logic (Yang 2022), where the constants are necessary for obtaining the desired expressivity. We say that a proof system is  $k$ -ary if each rule in the system has at most  $k$  assumptions. For usual inclusion atoms (with or without repetitions), there are 2-ary complete systems. While for inclusion atoms with Boolean constants, we show that there is no complete  $k$ -ary system that axiomatizes their implication problem. Nevertheless, for finite sets of these extended atoms, the implication problem remains PSPACE-complete, like for the repetition-free variants covered in (Casanova, Fagin, and Papadimitriou 1984).

The paper examines the limitations of some axiomatizations in the database literature and extends them to inclusion atoms used in team semantics. The team semantic framework also allows us to situate these atoms in richer languages with connectives, as in, e.g., (Yang 2022). Team semantics has close ties to knowledge representation as seen in recent KR-papers (Barlag et al. 2023; Sauerwald, Meier, and Kontinen 2025).

## 2 Main Results

We briefly recall the main results from the paper focused on the Boolean setting.

**Definition 1.** *We define rules I1-I6, B1 and B2, and schema B3 for inclusion atoms with Boolean constants. Let  $x$  be a sequence of constants  $\top$  and  $\perp$ .*

$$(I1) \quad p \subseteq p.$$

- (I2) If  $p \subseteq r$  and  $r \subseteq q$ , then  $p \subseteq q$ .  
(I3) If  $pqr \subseteq uvw$ , then  $prq \subseteq uvw$ ,  
provided that  $|p| = |u|$  and  $|q| = |v|$ .  
(I4) If  $pq \subseteq uv$ , then  $p \subseteq u$ .  
(I5) If  $pq \subseteq uv$ , then  $pqq \subseteq uvv$ .  
(I6) If  $p_1p_2 \subseteq q_1q_1$  and  $r \subseteq vp_1$ , then  $r \subseteq vp_2$ .  
(B1) If  $\top \subseteq \perp$ , then  $q \subseteq r$ .  
(B2) If  $p \subseteq q$ , then  $p\top \subseteq q\top$  and  $p\perp \subseteq q\perp$ .  
(B3) Let  $n := |p|$  and  $A \subseteq \{r \subseteq q \mid r_i \in \{p_i, \top, \perp\}, 1 \leq i \leq n\}$  be a minimal set such that for any  $x$  with consistent  $x \subseteq p$ , there is some  $r \subseteq q \in A$  such that  $x \subseteq r$  is consistent. If  $A$ , then  $p \subseteq q$ .

Rules I1-I4 are from (Casanova, Fagin, and Papadimitriou 1984), rules I5 and I6 are essentially from (Mitchell 1983), and rule B2 appears in (Yang 2022). Rule B1 captures that  $\top \subseteq \perp$  is not satisfied in any nonempty team.

Schema B3 deals with atoms containing Boolean constants, where we say that  $x_1 \dots x_n \subseteq r_1 \dots r_n$  is consistent if for all  $i, j \in \{1, \dots, n\}$ ,  $r_i \in \{\top, \perp\}$  implies  $r_i = x_i$ , and additionally, if  $r_i = r_j$ , then  $x_i = x_j$ . Now  $\top\perp \subseteq \top p_2$  and  $\top\top \subseteq p_2 p_2$  are consistent, while  $\top\perp \subseteq \perp p_2$  and  $\top\perp \subseteq p_2 p_2$  are not. The schema B3 has many instances: one example is that  $p_1 p_2 \subseteq q_1 q_2$  follows from  $\{\top p_2 \subseteq q_1 q_2, \perp p_2 \subseteq q_1 q_2\}$ . There are more involved instances where the number of assumptions required to apply B3 grows with the length of the sequences in the atoms, culminating in the following result.

**Theorem 1.** *There is no  $k$ -ary axiomatization of inclusion atoms with Boolean constants.*

The main results of the paper are the following novel completeness results for inclusion atoms in the Boolean setting.

**Theorem 2.**

- I1-I4 form a sound and complete proof system for repetition-free inclusion atoms.
- I1-I6 form a sound and complete proof system for inclusion atoms (with repetitions).
- I1-I6 together with B1-B3 form a sound and complete proof system for inclusion atoms with Boolean constants.

It follows that repetition-free inclusion atoms have the same complete system as in the FO-setting, thus we can use the corresponding FO result from (Casanova, Fagin, and Papadimitriou 1984) to conclude that the complexity of the implication problem for repetition-free inclusion atoms is PSPACE-complete also in the Boolean setting. We extend this complexity result to inclusion atoms with repetitions and Boolean constants.

**Theorem 3.** *Deciding whether  $\Sigma \models p \subseteq q$  is PSPACE-complete, provided that  $\Sigma \cup \{p \subseteq q\}$  is a finite set of inclusion atoms with (or without) Boolean constants.*

### 3 Conclusion

The paper addresses gaps in the literature regarding the axiomatization of the implication problem for variants of inclusion dependencies/atoms that are central in both database theory and team semantics. Complete proof systems and

the complexity of the decision problem for the considered variants are provided. Related work following the paper's publication includes the following two directions.

Inclusion atoms of the forms  $x \subseteq p$  and  $p \subseteq x$  are syntactically dual, and can be used as the main atoms in languages that capture all quasi upward and nonempty downward closed team properties, respectively (Häggblom 2026b). This shows the versatility of inclusion atoms with Boolean constants when situating them in richer team-based languages with connectives.

While we have focused on the Boolean setting, the alternative completeness proof for inclusion atoms also applies to the FO-setting, and gives better control (compared to the construction in (Casanova, Fagin, and Papadimitriou 1984)) over the degree to which some inclusion atom does not hold. This allows us to generalize it to prove completeness of an axiomatization for *quantity approximate* inclusion atoms (Häggblom 2026a), where we say that  $a \subseteq_n b$  holds if at most  $n$  values of  $a$  are absent from  $b$  in a team, connecting the results of the paper to the study of reasoning under uncertainty.

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