AGM Belief Revision, Semantically (Extended Abstract)

Faiq Miftakhul Falakh[1] , **Sebastian Rudolph**[1] , **Kai Sauerwald**[2]
[1]Computational Logic Group, Technische Universität Dresden
[2]Faculty of Mathematics and Computer Science, FernUniversität in Hagen

Abstract

The paper identifies a relational semantics for theory revision for various notions of bases in arbitrary Tarskian logics. Our result generalises the work by Delgrande, Woltran and Peppas on revision to this expressive setting, such that also logics with infinitely many interpretations are covered, as, e.g., in many predicate logics. Different notions of a base, e.g., belief sets, Hansson-style bases, finite sets, etc., can be uniformly described by the novel notion of a base logic, which we introduce. Furthermore, we identify a property of relations, min-retractivity, that gives rise to a representation theorem of AGM revision semantically in this general setting. Part of the representation theorem is an elegant method for encoding change operators. Moreover, we characterise those logics in which revision operators can be represented by a total preorder.

1 Introduction

The area of *belief change* studies formal models of how agents' beliefs change. A central operation considered therein is *AGM revision* (Alchourrón, Gärdenfors, and Makinson 1985), i.e., the process of incorporating new information while avoiding inconsistencies and respecting the principle of minimal change. One can implement a revision operator $\circ$ semantically by 'preferences' over interpretations. Roughly, the approach is as follows (Katsuno and Mendelzon 1992):

When changing agents' beliefs K upon receiving new information α, the new beliefs $K \circ \alpha$ are those whose models include the most 'preferred' models of α. (†)

However, with (†) one has to be careful, as not every notion of 'most preferred' leads to an AGM revision, whereby the latter is given by formal principles. Due to Katsuno and Mendelzon (1992) (KM) for propositional logic, for every K, the changes on K by $\circ$ are AGM revisions if and only if there is a total preorder such that one obtains $K \circ \alpha$ as in (†), i.e., informally, the following correspondence holds:

[Propositional Logic] AGM revision $\simeq$ total preorders. (‡)

The relationship given in (‡) is one of the most central insights in belief change theory. However, when leaving the safe ground of propositional logic, it has been observed that the relationship (‡) does not hold anymore, e.g., for Horn logics (Delgrande and Peppas 2015).

In the following, we review contributions by Falakh, Rudolph, and Sauerwald (2025) on relational semantics of AGM revision in the style of (†) beyond propositional logic.

2 Base Logics

When taking a closer look, the relationship given in (†) depends on many aspects:

(A) Representation of agents' beliefs K
(B) Representation of information α
(C) Representation of the revision result $K \circ \alpha$
(D) Notion of modelhood
(E) Type of 'preference' relations
(F) The principles to define AGM revision

We start with (A)–(D). There is consensus that agents' beliefs can be formalised by a means of logic, and also that one uses the same logic for representing K, α and $K \circ \alpha$. Model-theoretic logics, such as propositional logic or predicate logics, are often employed and provide a natural notion of modelhood that can be employed for (†). However, there exist different approaches on how the underlying logic is employed to form 'knowledge *bases*' for representing K and $K \circ \alpha$. To accommodate the plurality of approaches, we introduced the notion of *base logic*, in which one uses an abstract set of bases $\mathfrak{B}$.

Def. 1. *A* base logic *is a tuple* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$, *where*

- $(\mathcal{L}, \Omega, \models)$ *is a model-theoretic logic (with sentences $\mathcal{L}$, interpretations Ω, and satisfaction relation $\models$),*
- $\mathfrak{B} \subseteq \mathcal{P}(\mathcal{L})$ *is a family of sets of sentences, called* bases*,*
- $\Cup : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *is a binary operator over bases, called the* abstract union*, satisfying* $[\![\mathcal{B}_1 \Cup \mathcal{B}_2]\!] = [\![\mathcal{B}_1]\!] \cap [\![\mathcal{B}_2]\!]$.

Given a logic, typical choices of which kind of sets to pick as bases are *arbitrary sets* (as in the setting of base revision à la Hansson), *finite sets*, which are a natural representation in the context of computation, e.g. when computational properties or implementations are to be investigated. Another common choice is *belief sets*, i.e., the bases are deductively closed sets of sentences. One might also consider the setting where the bases are all *singleton sets*, which comes close to the setting investigated by KM (1992).

In the setting of base logics, we assume that representation of information – the (A)–(C) – is captured by $\mathfrak{B}$ uniformly, i.e., K, α, and $K \circ \alpha$ are all stemming from $\mathfrak{B}$. Formally, a *base change operator* for $\mathbb{B}$ is a function $\circ : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. For space reasons, we omit the obvious description of how AGM revision is transfered to the base logic setting (the (F)).

3 Representation Theorems and Encoding

Our article establishes a correspondence between classes of relations via (†), in the style of (‡), for the general setting of arbitrary base logics. Clearly, as (‡) does not hold for all logics, the corresponding class of relations $\leq$ cannot be total preorders. In the article, we define *min-retractivity*, which is a weakening of transitivity (Falakh, Rudolph, and Sauerwald 2025), demanding closure under certain refinements in $\leq$:

For all $\Gamma \in \mathfrak{B}$ and all $\omega, \omega' \in [\![\Gamma]\!]$ holds
$\omega \in \min([\![\Gamma]\!], \leq)$ and $\omega' \leq \omega$ imply $\omega' \in \min([\![\Gamma]\!], \leq)$.

(We let $[\![\Gamma]\!] = \{\omega{\in}\Omega \mid \omega{\models}\Gamma\}$.) The notion of min-retractivity is the right notion (the (E) from the last section) to capture AGM revision in base logics. Roughly expressed, we showed that the following relationship holds for all base logics:

[Basic Logic] AGM revision $\simeq$ total min-retractive relations

In certain special cases, e.g. for propositional logic, the property of being total and min-retractive coincides with being a total preorder. Because of that, the relationship given above subsumes (‡).

A central aspect of proving the correspondence between AGM revision and total min-retractive relations is to show a way to encode a given AGM revision operator into a total min-retractive relation. For that, we identified the following novel encoding scheme (for $\omega_1, \omega_2 \in \Omega$ and $\mathcal{K} \in \mathfrak{B}$)

$$\omega_1 \sqsubseteq^{\circ}_{\mathcal{K}} \omega_2 \text{ if for all } \Gamma \in \mathfrak{B} \text{ with } \omega_1, \omega_2 \in [\![\Gamma]\!] : \omega_2 \models \mathcal{K} \circ \Gamma \text{ implies } \omega_1 \models \mathcal{K} \circ \Gamma$$

that yields a relation $\sqsubseteq^{\circ}_{\mathcal{K}}$ for a base change operator $\circ$ and a knowledge base $\mathcal{K}$. Intuitively, $\omega_1 \sqsubseteq^{\circ}_{\mathcal{K}} \omega_2$ holds whenever there is no witness that ω_1 is not preferred over ω_2 (when performing revisions). We consider this encoding scheme as one of the most significant contributions of the paper.

4 Total-Preorder Representability

Because total preorders are considered as a basic structure for representing preferences in knowledge representation and beyond (Kelly 2013), there is theoretical interest in identifying the settings in which (‡) holds. We investigated total-preorder representability in two flavours:

(I) Identifying the class of AGM revision operators for which every revision operator can be represented by total preorders.

(II) Identifying those base logics in which every AGM revision operator can be represented by total preorders.

For (I), we adapted the principle of (Acyc) by Delgrande, Peppas, and Woltran (2018) to the setting of base logics:

(Acyc) For any base $\mathcal{K} \in \mathfrak{B}$ and any sequences of bases $\Gamma_0, \ldots, \Gamma_n \in \mathfrak{B}$ such that $[\![\Gamma_i \Cup (\mathcal{K} \circ \Gamma_{i\oplus 1})]\!] \neq \emptyset$ for each $0 \leq i \leq n$, where $\oplus$ denotes addition $\bmod (n+1)$, it holds $[\![\Gamma_0 \Cup (\mathcal{K} \circ \Gamma_n)]\!] \neq \emptyset$.

We showed that the (Acyc) principle also captures – in the very general setting of base logics – those AGM revision operators that can be represented by total preorders:

[Base Logic] AGM revision + Acyc $\simeq$ total preorders.

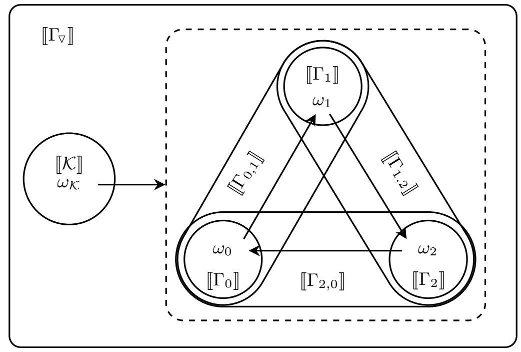

Figure 1: Graphical illustration of a critical loop. The areas within borders represent sets of models. The arrows depict a non-transitive relation between the interpretations; each arrow indicates that the interpretation at the start of the arrow is strictly preferred over the interpretation(s) at the arrow tip.

For (II), we introduced the notion of loop-free base logics. First, one defines a critical loop, which we present here in a simplified form for clarity. A critical loop consists of three bases $\Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0}$ whose model sets overlap as illustrated as in Figure 1 and bases $\Gamma_\triangledown$, Γ_0, Γ_1, Γ_2 and $\mathcal{K}$ are existing as illustrated. Important is that $\Gamma_\triangledown$ covers all models of $\Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0}$, and $\Gamma_\triangledown$ contains strictly more models. Moreover, Γ_0, Γ_1, Γ_2 ly in the respective intersections of $\Gamma_{0,1}, \Gamma_{1,2}$, and $\Gamma_{2,0}$. For $\mathcal{K}$ we demand $[\![\mathcal{K} \cup \Gamma_{i,i\oplus 1}]\!] = \emptyset$ for each $i \in \{0, 1, 2\}$ holds. As we show in the article, a critical loop permits the construction of a revision operator which can only be represented by a non-transitive relation. Figure 1 illustrates this relation. When there is no critical loop present in a base logic, we denote the base logic as *loop-free*. The base logics which are loop-free are *exactly* those with:

[Loop-free Basic Logic] AGM revision $\simeq$ total preorders.

Because logics with a disjunction connective are always loop-free, we obtain the following corollary, showing that for most well-known logics, like first-order predicate logic, the connection (‡) still holds.

[Disjunctive Basic Logic] AGM revision $\simeq$ total preorders.

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