Extending the Fluted Fragment with Transitivity and Counting

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Abstract

The fluted fragment is a fragment of first-order logic decidable for satisfiability, but which, unlike many such fragments, imposes no restriction on the number of variables or the use of negation. We highlight our recent results concerning the impact of adding both counting quantifiers and transitivity to the fluted fragment. The resulting formalism can be viewed as a multi-variable, higher-arity, non-guarded extension of certain description logics featuring number restrictions and transitive roles, but lacking role-inverses.

Introduction and contribution

We denote the satisfiability problem for any logical language $\mathcal{L}$ by $\text{Sat}(\mathcal{L})$. In knowledge representation and database theory an important reasoning problem is the query entailment problem over incomplete databases enriched by ontologies described in some logic $\mathcal{L}$. This problem reduces to unsatisfiability of the conjunction $\varphi \wedge \neg q$, where φ describes the ontology and q is a Boolean query. The quest for expressive logics able to formalize complex ontologies, but at the same time exhibiting acceptable computational behaviour, has attracted attention to a number of fragments of first-order logic, e.g. the guarded fragment, the guarded negation fragment, the unary negation fragment, besides the well-known family of description logics and various extensions of datalog. Since, in real-life ontologies, it is natural to find properties where transitivity interacts with counting, the search for logics with decidable (finite) query entailment allowing such an interaction has a strong practical motivation.

We highlight our recent results concerning the impact of adding both counting quantifiers and transitivity to the fluted fragment. The resulting formalism can be viewed as a multi-variable, higher-arity, non-guarded extension of certain description logics featuring number restrictions and transitive roles, but lacking role-inverses.

The *fluted fragment*, or $\mathcal{FL}$, originating in the work of Quine (1969), is a fragment of first-order logic in which, roughly speaking, the sequence of quantification of variables coincides with the order in which those variables appear as arguments of predicates. The *fluted fragment with counting*, or $\mathcal{FLC}$, is the extension of $\mathcal{FL}$ with the standard counting quantifiers $\exists_{[\leq M]}$, $\exists_{[\geq M]}$ and $\exists_{[=M]}$, where M is a (numeral denoting a) positive integer. The following sentence is in $\mathcal{FLC}$:

At most three lecturers introduce a professor to at least five students

$$\exists_{[\leq 3]} x_1 (\text{lectr}(x_1) \wedge \exists x_2 (\text{prof}(x_2) \wedge \exists_{[\geq 5]} x_3 (\text{std}(x_3) \wedge \text{intro}(x_1, x_2, x_3)))). \tag{1}$$

In the formal definition below, logical variables are taken from the sequence $\bar{x}_\omega = x_1, x_2, \ldots$, and all signatures are purely relational, i.e., there are no individual constants or function symbols. To establish the syntax of the fragment $\mathcal{FLC}$, *the fluted fragment with counting*, we first define the sets of formulas $\mathcal{FLC}^{[k]}$, for all $k \geq 0$, by simultaneous structural recursion as follows:

(i) any atom $p(x_\ell, \ldots, x_k)$, where $x_\ell, \ldots, x_k$ is a contiguous subsequence of $\bar{x}_\omega$ and p a (non-equality) predicate of arity $k - \ell + 1$, is in $\mathcal{FLC}^{[k]}$;

(ii) $\mathcal{FLC}^{[k]}$ is closed under Boolean combinations;

(iii) if φ is in $\mathcal{FLC}^{[k+1]}$, then $\exists x_{k+1}\, \varphi$ and $\forall x_{k+1}\, \varphi$ are in $\mathcal{FLC}^{[k]}$,

(iv) if φ is in $\mathcal{FLC}^{[k+1]}$ and M a non-negative integer, then $\exists_{[\leq M]} x_{k+1}\, \varphi$, $\exists_{[\geq M]} x_{k+1}\, \varphi$ and $\exists_{[=M]} x_{k+1}\, \varphi$ are in $\mathcal{FLC}^{[k]}$.

Define $\mathcal{FL}^{[k]}$ to be the fragment of $\mathcal{FLC}^{[k]}$ in which no counting quantifiers occur: i.e. Clause (iv) is dropped. Now let the fragment $\mathcal{FLC}$ be the union $\bigcup_{k\geq 0} \mathcal{FLC}^{[k]}$; similarly let $\mathcal{FL} = \bigcup_{k\geq 0} \mathcal{FL}^{[k]}$. By $\mathcal{FLC}$+nTr, we understand the same set of formulas as $\mathcal{FLC}$, but with n distinguished binary predicates required to be interpreted as transitive relations; similarly for $\mathcal{FL}$+nTr. Finally, define $\mathcal{FLC}^k$ to be the fragment of $\mathcal{FLC}$ in which at most the variables $x_1, \ldots, x_k$ appear (free or bound); and similarly for $\mathcal{FL}^k$, $\mathcal{FLC}^k$+nTr and $\mathcal{FL}^k$+nTr. For any logic $\mathcal{L}$ we write $\mathcal{L}_=$ to denote the extension of $\mathcal{L}$ in which equality is allowed.

Assuming that the arity of any predicate is fixed in advance, variables in fluted logic convey no information at all, and therefore can be omitted, similarly to the syntax employed in description logics. For example, formula (1) can be unambiguously reconstructed—up to a shift of variable indices—from: $\exists_{[\leq 3]}(\text{lectr} \wedge \exists(\text{prof} \wedge \exists_{[\geq 5]}(\text{std} \wedge \text{intro})))$.

It was shown in (Purdy 1996) that $\mathcal{FL}$ has the finite model property, whence its satisfiability problem is decidable. The satisfiability problem for $\mathcal{FL}^k$ is known to be in $(k{-}2)$-NExpTime for $k \geq 3$ (Pratt-Hartmann, Szwast, and Tendera 2019). This result extends to the fragment $\mathcal{FLC}$, though with a best-known upper complexity bound of $(k-1)$-NExpTime for k in the same range (Pratt-Hartmann 2021).

It is impossible, within the fluted fragment, to express the property of transitivity: in particular, the formula $\forall x_1 \forall x_2 \big(r(x_1, x_2) \to \forall x_3 \big(r(x_2, x_3) \to r(x_1, x_3)\big)\big)$ is not fluted, because the variable sequence in the atom $r(x_1, x_3)$ omits x_2. The question therefore arises as to whether the fragments $\mathcal{FL}$ or even $\mathcal{FLC}$ can be extended by declaring certain distinguished binary predicates to be interpreted as transitive relations. For $\mathcal{FL}$, this question was largely resolved in (Pratt-Hartmann and Tendera 2022). It was shown that $\mathcal{FL}_=$+1Tr lacks the finite model property, but has decidable satisfiability and finite satisfiability problems; the former problem, restricted to the k-variable sub-fragment is shown to be in k-NExpTime, and the latter in $(k{+}1)$-NExpTime. In the presence of two transitive relations but without equality, the fluted fragment loses the finite model property, with the decidability of satisfiability and finite satisfiability both unknown. With either two transitive relations and equality or three transitive relations, satisfiability and finite satisfiability are both undecidable. In (Pratt-Hartmann and Tendera 2023) we consider the combination of transitivity and counting. We show that, in the absence of equality, we may add a single transitive relation to the fragment $\mathcal{FLC}$ without losing the finite model property. In the presence of two transitive relations, however, the satisfiability and finite satisfiability problems for $\mathcal{FLC}$ are undecidable, even in the absence of equality. Table 1 presents the decidability frontier of the fragments discussed above and Table 2 gives an overview of the best known complexity bounds.

decidable	undecidable	reference
$\mathcal{FL}$+1Tr	$\mathcal{FL}^2$+3Tr	(i)
$\mathcal{FL}_=$+1Tr	$\mathcal{FL}^2_=$+2Tr	(i)
$\mathcal{FLC}$+1Tr	$\mathcal{FLC}^2$+2Tr	(ii)
?	$\mathcal{FLC}_=$+1Tr ?	open

Table 1: Decidability frontier. References: (i) Pratt-Hartmann and Tendera 2022, (ii) Pratt-Hartmann and Tendera 2023.

$\mathcal{L}$	lower bound	upper bound
$\mathcal{FL}^k$	$\lfloor k/2 \rfloor$-NExpTime	$(k{-}2)$-NExpTime
$\mathcal{FLC}^k$	—"—	$(k{-}1)$-NExpTime
$\mathcal{FL}^k$+1Tr	—"— ExpSpace $(k = 2, 3)$	k-NExpTime
$\mathcal{FLC}^k$+1Tr	—"—	$(k{+}1)$-NExpTime

Table 2: Complexity of the satisfiability problem for the k-variable $(k \geq 3)$ fragments without equality. All fragments listed in the Table enjoy the finite model property.

Relation to description logics

The basic description logic $\mathcal{ALC}$ is a notational variant of propositional multi-modal logic. Extensions of $\mathcal{ALC}$ are defined by allowing additional constructs, in particular: *number restrictions* (denoted $\mathcal{Q}$) corresponding to counting quantifiers as defined in this paper, *transitivity* of roles (denoted $\mathcal{S}$), role hierarchies ($\mathcal{H}$) corresponding to inclusions of binary relations, *nominals* ($\mathcal{O}$) and role 'inverses' ($\mathcal{I}$). The logic $\mathcal{SHOQ}$ constitutes a maximal description logic that can be embedded into $\mathcal{FLC}$ with transitive relations. It is known that the free combination of number restrictions and transitivity leads to undecidability of concept satisfiability even in the smaller logic $\mathcal{SHQ}$. Indeed, it was shown in (Kazakov, Sattler, and Zolin 2007) that just three roles (two of them transitive) are sufficient for undecidability. In response to these negative results, description logics standardly impose the syntactic restriction that transitive roles or their super-roles do not appear in number restrictions. With these restrictions, decidability is restored: concept satisfiability for $\mathcal{SHOQ}$ is ExpTime-complete, and for $\mathcal{SHOIQ}$ it is NExpTime-complete (Tobies 2001). On the other hand, there is no problem with allowing transitive relations to appear under number restrictions in description logics too weak to allow these roles to interact with each other. Thus, for example, concept satisfiability for $\mathcal{SOQ}$ is decidable, even without any additional syntactic restrictions. In the logic considered in (Pratt-Hartmann and Tendera 2023), only one transitive relation is available, but it is allowed to combine freely with other relations. We showed that this comparative freedom is, from the point of view of decidability of satisfiability, unproblematic as long as we remain within the confines of fluted logic—in effect, provided we have no access to role inverses.

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